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Jacobian Elliptic Functions sn, cn, dn

Function subprograms `RELFUN` and `DELFUN` calculate, for real argument x and real modulus k, the Jacobian elliptic functions $\mathrm{sn}(x,k)$, $\mathrm{cn}(x,k)$ and $\mathrm{dn}(x,k)$. The function $\mathrm{sn}(x,k)$ is the inverse of the elliptic integral of the first kind and is defined implicitly by

$$x = \int_0^{\mathrm{sn}(x,k)} \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} \qquad (k^2 \le 1).$$

The functions $\mathrm{cn}(x,k)$ and $\mathrm{dn}(x,k)$ are defined by

$$\mathrm{sn}^2(x,k)+\mathrm{cn}^2(x,k)=1, \quad k^2\mathrm{sn}^2(x,k)+\mathrm{dn}^2(x,k)=1, \quad \mathrm{cn}(0,k)=\mathrm{dn}(0,k)=1.$$

This definition can be extended for $k^2 > 1$ (Ref. 2) by means of

$$\mathrm{sn}(x,k)=k_1\mathrm{sn}(kx,k_1), \quad \mathrm{cn}(x,k)=\mathrm{dn}(kx,k_1), \quad \mathrm{dn}(x,k)=\mathrm{cn}(kx,k_1),$$

where $k_1 = 1/k$. For $k=0$ and $k^2=1$ these functions are elementary:

$$\mathrm{sn}(x,0)=\sin x, \quad \mathrm{cn}(x,0)=\cos x, \quad \mathrm{dn}(x,0)=1,$$

$$\mathrm{sn}(x,\pm1)=\tanh x, \quad \mathrm{cn}(x,\pm1)=\mathrm{dn}(x,\pm1)=\mathrm{sech}\, x.$$

Note that for $k^2 \neq 1$ the Jacobian elliptic functions are periodic (with respect to x) with period $4\mathrm{K}(k)$ if $k^2<1$ and $4k_1\mathrm{K}(k_1)$ if $k^2>1$, where $\mathrm{K}(k)$ is the complete elliptic integral of the first kind.

On CDC and Cray computers, the double-precision version `DELFUN` is not available.

Structure:

`SUBROUTINE` subprograms
User Entry Names: `RELFUN`, `DELFUN`
Obsolete User Entry Names: `ELFUN` $\equiv$ `RELFUN`

Usage:

For `t` = `R` (type `REAL`), `t` = `D` (type `DOUBLE PRECISION`),

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CALL tELFUN(X,AK2,SN,CN,DN)
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`X` (type according to `t`) The argument x.

`AK2` (type according to `t`) The value of k^2 (the square of the modulus).

`SN` (type according to `t`) On exit, `SN` $=\mathrm{sn}(\mathtt{X},k)$.

`CN` (type according to `t`) On exit, `CN` $=\mathrm{cn}(\mathtt{X},k)$.

`DN` (type according to `t`) On exit, `DN` $=\mathrm{dn}(\mathtt{X},k)$.

Method:

The sequence of the Gaussian arithmetic-geometric mean is used together with the Gauss transformation and, where appropriate, the Jacobi imaginary transformation. For values AK2 > 1, the reciprocal modulus transformation is performed. For details see **References**.

Accuracy:

RELFUN (except on CDC and Cray computers) has full single-precision accuracy. For most values of the arguments, DELFUN (and RELFUN on CDC and Cray computers) has an accuracy of approximately two significant digits less than the machine precision.

Restrictions:

$|x| \leq 3\mathrm{K}(k)$ $(0 < k^2 < 1)$, $|x| \leq 3k_1\mathrm{K}(k_1)$ $(k^2 > 1)$, where $\mathrm{K}(x)$ is the complete elliptic integral of the first kind. (See entries RELIKC and DELIKC in RELI1C (C347)).

References:

1. M. Abramowitz and I.A. Stegun, ed., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Sections 16.12 and 17.6, 9th printing with corrections, (Dover, New York 1972).
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3. L.V. King, On the dirct numerical calculation of elliptic functions and integrals, Cambridge Univ. Press (1924) 32.
4. D.J. Hofsommer and R.P. van de Riet, On the numerical calculation of elliptic integrals of the first and second kind and the elliptic functions of Jacobi, Numer. Math. **5** (1963) 291–302.
5. P.F. Byrd and M.D. Friedman, Handbook of elliptic integrals for engineers and scientists, 2nd ed., Springer-Verlag Berlin (1971).

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