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Elliptic Integrals of First, Second, and Third Kind

Function subprograms `RELI1`, `RELI2`, `RELI3` and `DELI1`, `DELI2`, `DELI3` calculate, for real argument x, the elliptic integrals of the first, second and third kind, respectively.

On CDC and Cray computers, the double-precision versions `DELI1`, `DELI2` and `DELI3` are not available.

Mainly for reasons of numerical stability, the algorithms are based on the following definitions:

First kind:

$$\mathbf{F}_1(x,k') \;=\; \int_0^x \frac{d\xi}{\sqrt{(1+\xi^2)(1+k'^2\xi^2)}} \qquad (k'^2 \geq 0).$$

Second kind:

$$\mathbf{F}_2(x,k',a,b) \;=\; \int_0^x \frac{a+b\xi^2}{(1+\xi^2)\sqrt{(1+\xi^2)(1+k'^2\xi^2)}}\,d\xi \qquad (k'^2 \geq 0).$$

Third kind:

$$\mathbf{F}_3(x,k',p) \;=\; \int_0^x \frac{1+\xi^2}{(1+p\xi^2)\sqrt{(1+\xi^2)(1+k'^2\xi^2)}}\,d\xi \qquad (k'^2 \geq 0,\; px^2 \neq -1).$$

Note that $\mathbf{F}_1(x,k') = \mathbf{F}_2(x,k',1,1) = \mathbf{F}_3(x,k',1)$. For $p < 0$, the integral $\mathbf{F}_3$ is defined by its principal value.

For the integral of the second kind, a special entry-mode argument is provided which allows $\mathbf{F}_2(x,k',a,b)$ to be calculated when $k'^2 < 0$, i.e. when k' is imaginary.

Other common definitions of the elliptic integrals and their relations to $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ are listed here for convenience ($k^2 + k'^2 = 1$):

First kind:

$$F(k,\phi) \;=\; \int_0^\phi \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} \;=\; \mathbf{F}_1(\tan\phi,k') \qquad (|\phi| \leq \pi/2,\; |k| < 1),$$

$$\widehat{F}(y,k) \;=\; \int_0^y \frac{d\eta}{\sqrt{(1-\eta^2)(1-k^2\eta^2)}} \;=\; \mathbf{F}_1\left(y/\sqrt{1-y^2},k'\right) \qquad (|y| < 1,\; |k| < 1).$$

Second kind:

$$E(k,\phi) \;=\; \int_0^\phi \sqrt{1-k^2\sin^2\psi}\,d\psi \;=\; \mathbf{F}_2(\tan\phi,k',1,k'^2) \qquad (|\phi| \leq \pi/2,\; |k| \leq 1),$$

$$\widehat{E}(y,k) \;=\; \int_0^y \sqrt{\frac{1-k^2\eta^2}{1-\eta^2}}\,d\eta \;=\; \mathbf{F}_2\left(y/\sqrt{1-y^2},k',1,k'^2\right) \qquad (|y| < 1,\; |k| \leq 1).$$

Third kind:

$$\Pi(\phi,h,k) = \int_0^{\phi} \frac{d\psi}{(1+h\sin^2\psi)\sqrt{1-k^2\sin^2\psi}} = \mathbf{F}_3(\tan\phi,k',h+1) \quad (|\phi|\le\pi/2,\ |k|<1),$$

$$\widehat{\Pi}(y,h,k) = \int_0^{y} \frac{d\eta}{(1+h\eta^2)\sqrt{(1-\eta^2)(1-k^2\eta^2)}} = \mathbf{F}_3\left(y/\sqrt{1-y^2},k',h+1\right) \quad (|y|<1,\ |k|<1).$$

Structure:

`FUNCTION` subprograms
User Entry Names: `RELI1`, `RELI2`, `RELI3`, `DELI1`, `DELI2`, `DELI3`
Files Referenced: `Unit 6`
External References: `ASINH` (B102), `DASINH` (B102), `MTLMTR` (N002), `ABEND` (Z035)

Usage:

In any arithmetic expression, with `AKP` $= k'$,

`RELI1(X,AKP)`	or	`DELI1(X,AKP)`	has the value	$\mathbf{F}_1(\mathtt{X},k')$,
`RELI2(X,AKP,A,B,MODE)`	or	`DELI2(X,AKP,A,B,MODE)`	has the value	$\mathbf{F}_2(\mathtt{X},k'',\mathtt{A},\mathtt{B})$,
`RELI3(X,AKP,P)`	or	`DELI3(X,AKP,P)`	has the value	$\mathbf{F}_3(\mathtt{X},k',\mathtt{P})$,

where `RELI1`, `RELI2`, `RELI3` are of type `REAL`, where `DELI1`, `DELI2`, `DELI3` are of type `DOUBLE PRECISION`, and `X`, `AKP`, `A`, `B` and `P` have the same type as the function name. `MODE` is of type `INTEGER`.

The notation k'' indicates that, when calling `RELI2` or `DELI2`, the parameters `AKP` and `MODE` must be set as follows:
If $k'^2 > 0$: `MODE` $= +1$ and `AKP` $= k'$,
if $k'^2 < 0$: `MODE` $= -1$ and `AKP` $=$ Im $k' = -ik'$ (real).

Method:

The evaluation of $\mathbf{F}_1$ and $\mathbf{F}_2$ is based on the Landen transformation, that of $\mathbf{F}_3$ on the Bartky transformation. $\mathbf{F}_2$ for $k'^2 < 0$ is calculated by using a transformation of the arguments. See Ref. 1 and 2 for details.

Accuracy:

The `REAL` functions (except on CDC and Cray computers) have full single-precision accuracy. The `REAL` functions on CDC and Cray, and the `DOUBLE PRECISION` functions on all computers have an accuracy approximately two significant digits less than the machine precision.

Restrictions:

1. `RELI2` and `DELI2`: `AKP*X**2` $<$ `1` if `MODE` $= -1$.
2. `RELI2` and `DELI2`: `MODE` $= +1$ or -1.
3. `RELI3` and `DELI3`: `P*X**2` $\neq -1$.

Error handling:

Error `C346.1`: Restriction 1 is not satisfied.
Error `C346.2`: Restriction 2 is not satisfied.
Error `C346.3`: Restriction 3 is not satisfied.
In all cases, the function value is set equal to zero, and a message is written on `Unit 6`, unless subroutine `MTLSET` (N002) has been called.

Source:

The subprograms are based on the Algol60 procedures *el1, el2* in Ref. 1 and *el3* in Ref. 2.

References:

1. R. Bulirsch, Numerical calculation of elliptic integrals and elliptic functions, Numer. Math. **7** (1965) 78–90.
2. R. Bulirsch, Numerical calculation of elliptic integrals and elliptic functions III, Numer. Math. **13** (1969) 305–315.

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