RELI1C	CERN Program Library	C347

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Language : Fortran **Revised:**

Complete Elliptic Integrals of First, Second, and Third Kind

Function subprograms `RELI1C`, `RELI2C`, `RELI3C` and `DELI1C`, `DELI2C`, `DELI3C` calculate the complete elliptic integrals of the first, second and third kind, respectively.

Function subprograms `RELIGC` and `DELIGC` calculate a general complete elliptic integral.

Function subprograms `RELIKC`, `RELIEC` and `DELIKC`, `DELIEC` calculate the complete elliptic integrals $\mathrm{K}(k)$ and $\mathrm{E}(k)$.

On CDC and Cray computers, the double-precision versions `DELI1C` etc. are not available.

Mainly for reasons of numerical stability, the algorithms are based on the following definitions:

First kind:

$$\mathbf{F}_1^*(k') = \int_0^\infty \frac{d\xi}{\sqrt{(1+\xi^2)(1+k'^2\xi^2)}} \qquad (k'^2 > 0).$$

Second kind:

$$\mathbf{F}_2^*(k',a,b) = \int_0^\infty \frac{a+b\xi^2}{(1+\xi^2)\sqrt{(1+\xi^2)(1+k'^2\xi^2)}}\,d\xi \qquad (k'^2 > 0).$$

Third kind:

$$\mathbf{F}_3^*(k',p) = \int_0^\infty \frac{1+\xi^2}{(1+p\xi^2)\sqrt{(1+\xi^2)(1+k'^2\xi^2)}}\,d\xi \qquad (k'^2 > 0,\ p \neq 0).$$

Note that $\mathbf{F}_1^*(k') = \mathbf{F}_2^*(k',1,1) = \mathbf{F}_3^*(k',1)$. For $p < 0$, the integral $\mathbf{F}_3^*$ is defined by its principal value.

The general integral:

$$\begin{aligned}\mathbf{G}(k',p,a,b) &= \int_0^\infty \frac{a+b\xi^2}{(1+p\xi^2)\sqrt{(1+\xi^2)(1+k'^2\xi^2)}}\,d\xi \\ &= \int_0^{\pi/2} \frac{a\cos^2\phi + b\sin^2\phi}{\cos^2\phi + p\sin^2\phi}\,\frac{d\phi}{\sqrt{\cos^2\phi + k'^2\sin^2\phi}} \qquad (k'^2 > 0).\end{aligned}$$

For $p < 0$, this integral is defined by its principal value. See **Notes** for special cases.

The functions K(k) and E(k):

$$\begin{aligned}\mathrm{K}(k) &= \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} \qquad (|k| < 1), \\ \mathrm{E}(k) &= \int_0^{\pi/2} \sqrt{1-k^2\sin^2\psi}\,d\psi \qquad (|k| \le 1).\end{aligned}$$

Other common definitions of the complete elliptic integrals and their relations to $\mathbf{F}_1^*$, $\mathbf{F}_2^*$, $\mathbf{F}_3^*$ are listed here for convenience ($k^2 + k'^2 = 1$):

First kind:

$$F(k,\pi/2) = \mathrm{K}(k) = \mathbf{F}_1^*(k') \qquad (|k|<1),$$

$$\widehat{F}(1,k) = \int_0^1 \frac{d\eta}{\sqrt{(1-\eta^2)(1-k^2\eta^2)}} = \mathbf{F}_1^*(k') \qquad (|k|<1).$$

Second kind:

$$E(k,\pi/2) = \mathrm{E}(k) = \mathbf{F}_2^*(k',1,k'^2) \qquad (|k|\le 1),$$

$$\widehat{E}(1,k) = \int_0^1 \sqrt{\frac{1-k^2\eta^2}{1-\eta^2}}\,d\eta = \mathbf{F}_2^*(k',1,k'^2) \qquad (|k|\le 1).$$

Third kind:

$$\Pi(\pi/2,h,k) = \int_0^{\pi/2} \frac{d\psi}{(1+h\sin^2\psi)\sqrt{1-k^2\sin^2\psi}} = \mathbf{F}_3^*(k',h+1) \qquad (|k|<1),$$

$$\widehat{\Pi}(1,h,k) = \int_0^1 \frac{d\eta}{(1+h\eta^2)\sqrt{(1-\eta^2)(1-k^2\eta^2)}} = \mathbf{F}_3^*(k',h+1) \qquad (|k|<1).$$

Structure:

`FUNCTION` subprograms
User Entry Names: `RELI1C`, `RELI2C`, `RELI3C`, `RELIGC`, `RELIKC`, `RELIEC`
`DELI1C`, `DELI2C`, `DELI3C`, `DELIGC`, `DELIKC`, `DELIEC`
Obsolete User Entry Names: `ELLICK` $\equiv$ `RELIKC`, `ELLICE` $\equiv$ `RELIEC`,
`DELLIK` $\equiv$ `DELIKC`, `DELLIE` $\equiv$ `DELIEC`
Files Referenced: `Unit 6`
External References: `MTLMTR` (N002), `ABEND` (Z035)

Usage:

In any arithmetic expression, with `AK` $= k$ and `AKP` $= k'$,

`RELI1C(AKP)`	or	`DELI1C(AKP)`	has the value	$\mathbf{F}_1^*(k')$,
`RELI2C(AKP,A,B)`	or	`DELI2C(AKP,A,B)`	has the value	$\mathbf{F}_2^*(k',\mathtt{A},\mathtt{B})$,
`RELI3C(AKP,AK2,P)`	or	`DELI3C(AKP,AK2,P)`	has the value	$\mathbf{F}_3^*(k',\mathtt{P})$,
`RELIGC(AKP,P,A,B)`	or	`DELIGC(AKP,P,A,B)`	has the value	$\mathbf{G}(k',\mathtt{P},\mathtt{A},\mathtt{B})$,
`RELIKC(AK)`	or	`DELIKC(AK)`	has the value	$\mathrm{K}(k)$,
`RELIEC(AK)`	or	`DELIEC(AK)`	has the value	$\mathrm{E}(k)$,

where `RELI1C` etc are of type `REAL`, `DELI1C` etc are of type `DOUBLE PRECISION`, and `AKP`, `AK`, `AK2`, `A`, `B` and `P` have the same type as the function name.

The redundant parameter `AK2` in `RELI3C` and `DELI3C` permits improved accuracy when k^2 is small, i.e. $k' \approx 1$. In this case, `AK2` $= k^2$ should be calculated using higher-precision arithmetic and then truncated before calling the subprogram.

Method:

The evaluation of $\mathbf{F}_1^*$, $\mathbf{F}_2^*$, $\mathbf{F}_3^*$ is based on the Landen transformation, that of $\mathbf{G}$ on the Bartky transformation. For details, see Ref. 1–3. For $\mathrm{K}(k)$ and $\mathrm{E}(k)$ Chebyshev approximations are used (see Ref. 4).

Accuracy:

The `REAL` functions (except on CDC and Cray computers) have full single-precision accuracy. The `REAL` functions on CDC and Cray, and the `DOUBLE PRECISION` functions on all computers have an accuracy approximately two significant digits less than the machine precision.

Restrictions:

1. `RELI1C` and `DELI1C`: `AKP` $\neq$ `0`.
2. `RELI2C` and `DELI2C`: `AKP` $\neq$ `0` or `AKP` $=$ `0` and `B` $=$ `0`.
3. `RELI3C` and `DELI3C`: `AKP*P` $\neq$ `0`.
4. `RELIGC` and `DELIGC`: `AKP` $\neq$ `0`.
5. `RELIKC` and `DELIKC`: $|$`AK`$| \leq$ `1`, `RELIEC` and `DELIEC`: $|$`AK`$| <$ `1`.

Error handling:

Error `C347.1`: Restriction 1 is not satisfied.
Error `C347.2`: Restriction 2 is not satisfied.
Error `C347.3`: Restriction 3 is not satisfied.
Error `C347.4`: Restriction 4 is not satisfied.
Error `C347.5`: Restriction 5 is not satisfied.
In all cases, the function value is set equal to zero, and a message is written on `Unit 6`, unless subroutine `MTLSET` (N002) has been called.

Notes:

Every linear combination of the three special complete elliptic integrals $\mathrm{K}(k)$, $\mathrm{E}(k)$, $\Pi(h,k)$ may be expressed in terms of $\mathbf{G}(k',p,a,b)$:

$$\begin{aligned}\lambda\mathrm{K}(k)+\mu\mathrm{E}(k) &= \mathbf{G}(k',1,\lambda+\mu,\lambda+\mu k'^2)\\ \lambda\mathrm{K}(k)+\mu\Pi(h,k) &= \mathbf{G}(k',h+1,\lambda+\mu,\lambda(h+1)+\mu)\end{aligned}$$

Special examples are

$$\begin{aligned}\mathrm{K}(k) &= \mathbf{G}(k',1,1,1),\\ \mathrm{E}(k) &= \mathbf{G}(k',1,1,k'^2)\\ (\mathrm{K}(k)-\mathrm{E}(k))/k^2 &= \mathbf{G}(k',1,0,1),\\ (\mathrm{K}(k)-k'^2\mathrm{E}(k))/k^2 &= \mathbf{G}(k',1,1,0),\\ \Pi(h,k) &= \mathbf{G}(k',h+1,1,1),\\ (\mathrm{K}(k)-\Pi(h,k))/h &= \mathbf{G}(k',h+1,0,1),\end{aligned}$$

If $ab \geq 0$ then $\mathbf{G}$ will evaluate any linear combination of $\mathrm{K}(k)$, $\mathrm{E}(k)$, $\Pi(h,k)$ without cancellation (such as would occur, for example, if $(\mathrm{K}(k){-}\mathrm{E}(k))/k^2$ were to be computed from values of $\mathrm{K}(k)$ and $\mathrm{E}(k)$ which had been computed separately.
Other functions which can be represented by $\mathbf{G}$ are the Jacobian Zeta function $\mathbf{Z}(\Phi,k)$ and the Heuman Lambda function $\Lambda_0(\Phi,k)$ (see Ref. 5):

$$\begin{aligned}\mathbf{Z}(\Phi,k) &= k^2\frac{\sin\Phi\cos\Phi}{\mathrm{K}(k)}\mathbf{G}(k',q,0,\sqrt{q}) && (q=\cos^2\Phi+k'^2\sin^2\Phi)\\ \Lambda_0(\Phi,k) &= \frac{2}{\pi}\sqrt{q}\sin\Phi\,\mathbf{G}(k',q,1,k'^2) && (q=1+k^2\tan^2\Phi).\end{aligned}$$

(Quoted from Ref. 3, slightly modified).

Source:

The subprograms for $\mathbf{F}_1^*$, $\mathbf{F}_2^*$ are based on the Algol60 procedures *cel1, cel2* in Ref. 1, those for $\mathbf{F}_3^*$ on *cel3* in Ref. 2, and those for $\mathbf{G}$ on *cel* in Ref. 3.

References:

1. R. Bulirsch, Numerical calculation of elliptic integrals and elliptic functions, Numer. Math. **7** (1965) 78–90.
2. R. Bulirsch, Numerical calculation of elliptic integrals and elliptic functions II, Numer. Math. **7** (1965) 353–354.
3. R. Bulirsch, Numerical calculation of elliptic integrals and elliptic functions III, Numer. Math. **13** (1969) 305–315.
4. W.J. Cody, Chebyshev approximations for the complete elliptic integrals K and E, Math. Comp. **19** (1965) 105–112.
5. P.F. Byrd and M.D. Friedman, Handbook of elliptic integrals for engineers and scientists, 2nd ed., Springer-Verlag Berlin (1971) 33–37.

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