Author(s) : G.A. Erskine
Submitter : K.S. Kölbig
Language : Fortran

Library: MATHLIB
Submitted: 07.06.1992
Revised:

Jacobian Theta Functions

Function subprograms RTHETA and DTHETA calculate the Jacobian theta functions

$$\begin{aligned}
\vartheta_0(x,q) &= 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2n\pi x,\\
\vartheta_1(x,q) &= 2\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}\sin(2n+1)\pi x,\\
\vartheta_2(x,q) &= 2\sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2}\cos(2n+1)\pi x,\\
\vartheta_3(x,q) &= 1+2\sum_{n=1}^{\infty} q^{n^2}\cos 2n\pi x,\\
\vartheta_4(x,q) &= \vartheta_0(x,q),
\end{aligned}$$

for real arguments x and $0 \le q < 1$. $\vartheta_1(x+\frac{1}{2},1)$ and $\vartheta_2(x,1)$ are undefined if x is an integer; otherwise $\vartheta_k(x,1)=0,\ k=1,2,3,4$.

Note that several conflicting definitions of these functions occur in the literature. In particular, the argument in the trigonometric terms is often defined to be x instead of πx.

On CDC and Cray computers, the double-precision version DTHETA is not available.

Structure:

FUNCTION subprogram
User Entry Names: RTHETA, DTHETA
Files Referenced: Unit 6
External References: MTLMTR (N002), ABEND (Z035)

Usage:

In any arithmetic expression,

RTHETA(K,X,Q) or DTHETA(K,X,Q) has the value $\vartheta_{\mathrm{K}}(\mathrm{X},\mathrm{Q})$,

where RTHETA is of type REAL, DTHETA is of type DOUBLE PRECISION, X and Q are of the same type as the function name, and K is of type INTEGER.

Method:

If t $(0 \le t \le \frac{1}{2})$ differs from x or $-x$ by an integer, it follows from the periodicity and symmetry properties of the functions that $\vartheta_1(x,q)=\pm\vartheta_1(t,q)$ and $\vartheta_3(x,q)=\vartheta_3(t,q)$. In a region for which the approximation is sufficiently accurate, ϑ_1 is set equal to the first $(n=0)$ term of the transformed series

$$\vartheta_1(t,q) = 2(\lambda/\pi)^{1/2}e^{-\lambda t^2}\sum_{n=0}^{\infty}(-1)^n e^{-\lambda(n+\frac{1}{2})^2}\sinh(2n+1)\lambda t,$$

and ϑ_3 is set equal to the first two (i.e. $n \leq 1$) terms of

$$\vartheta_3(t,q) \;=\; (\lambda/\pi)^{1/2} e^{-\lambda t^2} \left(1 + 2\sum_{n=1}^{\infty} e^{-\lambda n^2} \cosh 2n\lambda t\right),$$

where $\lambda = \pi^2/|\ln q|$. Otherwise the trigonometric series for $\vartheta_1(t,q)$ and $\vartheta_3(t,q)$ are used.
For all x, ϑ_0 and ϑ_2 are computed from $\vartheta_0(x,q) = \vartheta_3(\frac{1}{2} - |x|, q)$, $\vartheta_2(x,q) = \vartheta_1(\frac{1}{2} - |x|, q)$.

Restrictions:

1. $\mathtt{0} \leq \mathtt{Q} \leq \mathtt{1}$.
2. $\mathtt{K} = \mathtt{0}, \mathtt{1}, \mathtt{2}, \mathtt{3}, \mathtt{4}$.
3. If $\mathtt{Q} = \mathtt{1}$ and $\mathtt{K} = \mathtt{1}$, $\mathtt{X} - \frac{1}{2}$ must not be an integer.
   If $\mathtt{Q} = \mathtt{1}$ and $\mathtt{K} = \mathtt{2}$, `X` must not be an integer.

Error handling:

Error `C349.1`: Restriction 1 is not satisfied.
Error `C349.2`: Restriction 2 is not satisfied.
Error `C349.3`: Restriction 3 is not satisfied.
In all cases, the function value is set equal to zero, and a message is written on `Unit 6`, unless subroutine `MTLSET` (N002) has been called.

Accuracy:

For `DTHETA` (and for `RTHETA` on CDC and Cray computers), the error when `Q` is less than approximately `0.9` does not exceed two decimal digits in the last place. For larger values of `Q` (provided the computed result is non-zero), the error is at worst comparable in magnitude to the mathematical error which would be caused by one-bit rounding errors in the arguments `X` and `Q`.

On computers other than CDC and Cray, non-zero values of `RTHETA` have full machine accuracy.

Notes:

Successive references using the same value of `Q` are executed faster than those in which `Q` changes.

Many functional relations, including relations between the theta functions and the Jacobian elliptic functions, are given in Refs. 1–4.

References:

1. W. Magnus, F. Oberhettinger and R.P. Soni, Formulas and theorems for the special functions of mathematical physics, Springer-Verlag Berlin (1966) 371–377.

2. F. Tölke, Praktische Funktionenlehre, Bd. II, Springer-Verlag Berlin (1966) 1–38.

3. P.F. Byrd and M.D. Friedman, Handbook of elliptic integrals for engineers and scientists, 2nd Edition, Springer-Verlag Berlin (1971) 315–320.

4. E.T. Whittaker and G.N. Watson, A course of modern analysis, 4th Edition, Cambridge University Press, Cambridge (1946) Chapter 21.

•